第 53 卷 第 3 期　　气 象 学 报　　Vol. 53, No. 3
1995 年 8 月　　ACTA METEOROLOGICA SINICA　　Aug. , 1995

准地转动量近似下风速切变线上的波动*

李子良　　万军

(中国民航飞行学院，广汉，618307)　(成都气象学院，成都，610041)

摘　要

利用准地转动量近似下的斜压两层大气的非线性方程组及非线性边界条件，研究了切变线上波动的稳定性，导出了切变线上孤立波解。较好地解释了切变线上易产生低涡等天气系统现象。

关键词：准地转动量近似，小参数方法，KDV 方程，孤立波。

1　前言

风速切变线是一种重要的天气系统，我国气象工作者早就发现，切变线上可有低涡生成，低涡沿切变线东移并可造成降水。有人曾对准水平无辐散的正压线性波动的稳定性进行讨论并以此来解释切变线上低涡的形成[1]。但切变线上低涡的形成本质上是一个非线性问题，非线性的作用不能忽略。本文采用准地转动量近似和小参数方法[2]，研究了斜压两层大气中水平风速切变线上的非线性方程组及非线性边界条件，得到了描述非线性和色散性的 KDV 方程，求得了切变线上的孤立波解，以此来说明水平风速切变对低涡形成所起的作用。

2　运动方程及边界条件

设在斜压两层大气中存在一水平风速切变，风速切变线的分界面设为 $y=\eta(x,t)$，其初始位置设为 x 轴。切变线的南侧和北侧的基本风速分别为 $\bar{u}_1,\bar{u}_2$，均为常数，并设基本流场在 y 方向伸展为无穷远。则斜压两层大气中[4]水平风速切变线上的非线性方程组为：

$$
\begin{aligned}
&\frac{\partial u_j}{\partial t}+u_j\frac{\partial u_j}{\partial x}+v_j\frac{\partial u_j}{\partial y}-fv_j=-\frac{\partial\varphi_j}{\partial x}\\
&\frac{\partial v_j}{\partial t}+u_j\frac{\partial v_j}{\partial x}+v_j\frac{\partial v_j}{\partial y}+fu_j=-\frac{\partial\varphi_j}{\partial y}\\
&\frac{\partial\varphi_j}{\partial t}+u_j\frac{\partial\varphi_j}{\partial x}+v_j\frac{\partial\varphi_j}{\partial y}=-\varphi_j\left(\frac{\partial u_j}{\partial x}+\frac{\partial v_j}{\partial y}\right)
\end{aligned}
\tag{1}
$$

式中 $j=1,2$，u_j，v_j 和 φ_j 分别表示 x，y 方向的风速分量和重力位势，f 为科氏参数，采用 β 平面近似，取 $f=f_0+\beta_0 y$，其中 f_0，β_0 均为常数。

* 1993 年 7 月 9 日收到原稿，1994 年 3 月 21 日收到修改稿。

由于准地转动量近似保留了倾斜效应，使得水平切变向涡旋扭转，水平涡度向垂直涡度扭转[3]，由此求得的非线性波具有充分的涡旋特征；同时准地转动量近似不仅保留了全部的非线性项，而且可以滤去快波[2]，在数学处理上简便得多，因此我们采用准地转动量近似。同时，平流量（含 f_0 的项）用实际场而被平流量（含 β_0 的项）用地转量代替（因含 β_0 的项已具有充分的非地转特征）。用下标"g"表示地转量，有：

$$
\begin{aligned}
&(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}+v_j\frac{\partial}{\partial y})u_{gj}-f_0v_j-\beta_0yv_{gj}=-\frac{\partial\varphi_j}{\partial x}\\
&(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}+v_j\frac{\partial}{\partial y})v_{gj}+f_0u_j+\beta_0yu_{gj}=-\frac{\partial\varphi_j}{\partial y}\\
&(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}+v_j\frac{\partial}{\partial y})\varphi_j=-\varphi_j(\frac{\partial u_j}{\partial x}+\frac{\partial v_j}{\partial y})
\end{aligned}
\tag{2}
$$

式中 $j=1,2$，u_{gj}，v_{gj} 分别表示 x，y 方向的地转风速分量。

式(2)即为准地转动量近似下描述涡旋运动的切变线上的非线性运动方程组。

风速切变线上的非线性动力学边界条件为：当 $y=\eta(x,t)$ 时

$$
(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x})(\varphi_1'-\varphi_2')+v_j'\frac{\partial(\bar{\varphi}_1-\bar{\varphi}_2)}{\partial y}=0 \tag{3}
$$

式中 v_j' 表示 y 方向的扰动场，φ_1'，φ_2' 分别表示切变线南、北两侧的重力位势扰动量。取自然边界条件为：

当

$$
\begin{aligned}
&y\to-\infty\ 时， && \varphi_1'\to 0\\
&当\ y\to+\infty\ 时， && \varphi_2'\to 0
\end{aligned}
\tag{4}
$$

3 各级近似方程

式(2)中

$$
\varphi_j=\varphi_0+\varphi_j' \qquad \varphi_j=g^*H_j=g^*H_0+g^*H_j'
$$

$$
g^*=g\frac{\rho-\rho'}{\rho}
$$

ρ' 和 ρ 分别为上层和下层流体密度，引入如下特征量：

$$
\begin{aligned}
&t=\tau(t),\triangle\varphi_j=f_0LU \quad u_j,v_j=U(u_j,v_j)\\
&x,y=L(x,y),Coj=C_0C
\end{aligned}
\tag{5}
$$

我们研究的是天气尺度运动，所以取 $L\sim1000\text{km}$，$U=10\text{ms}^{-1}$，$f_0\sim10^{-4}\text{s}^{-1}$，$\beta_0\sim10^{-11}\text{m}^{-1}\text{s}^{-1}$，将式(5)代入式(2)有

$$
\begin{aligned}
&\varepsilon_i(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}+v_j\frac{\partial}{\partial y})u_{gj}-v_j-\varepsilon_iv_{gj}=-\frac{\partial\varphi_j}{\partial x}\\
&\varepsilon_i(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}+v_j\frac{\partial}{\partial y})v_{gj}+u_j+\varepsilon_iu_{gj}=-\frac{\partial\varphi_j}{\partial y}\\
&(\frac{\partial}{\partial t}+u_j\frac{\partial}{\partial x}+v_j\frac{\partial}{\partial y})\varphi_j=-\lambda^{-2}(\frac{\partial u_j}{\partial x}+\frac{\partial v_j}{\partial y})
\end{aligned}
\tag{6}
$$

式中 $\lambda^2 = L^2/L_0^2, L_0^2 = C_0^2/f_0^2, C_0^2 = g^* H_0$。对于两层斜压大气,$C_0^2 = g^* H_0 \sim 50ms^{-1}$,所以有 $L_0 \sim 500km, \lambda \sim 2$。

对于无因次方程式(6),式中只有两个参数:参数 λ^{-2} 和基别尔参数 $\varepsilon_i = \dfrac{1}{f_0\tau}$,为了使切变线上的波动具有充分的非线性特征,式中参数 $\varepsilon_i \geqslant 1$,因而式中只有唯一的小参数 λ^{-2} 远小于 1,所以在以下讨论中及采用小参数展开方法中,将用 $\varepsilon = \lambda^{-2}$ 作为小参数。先用小参数 ε 将基本流场和扰动流场区别开来,

$$
\begin{aligned}
&u_j = \bar{u}_j + \varepsilon u_j' \qquad v_j = \varepsilon v_j' \quad \varphi_j = \bar{\varphi}_j(y) + \varepsilon \varphi_j' \\
&u_{gj} = \bar{u}_{gj} + u_{gj}' \qquad v_{gj} = \varepsilon v_{gj}'
\end{aligned} \tag{7}
$$

式中 $\bar{u}_j$、$\bar{u}_{gj}$ 为基本流场,u_j'、v_j' 和 u_{gj}'、v_{gj}' 为扰动流场,$\bar{\varphi}_j(y)$ 为基本的位势场,φ_j' 为扰动位势场。且 $\bar{u}_{gj}$、u_{gj}' 满足地转关系

$$
\begin{aligned}
&\bar{u}_{gj} = -\frac{1}{f_0}\frac{\partial\bar{\varphi}_j}{\partial y} \\
&u_{gj}' = -\frac{1}{f_0}\frac{\partial\varphi_j'}{\partial y} \quad v_{gj}' = \frac{1}{f_0}\frac{\partial\varphi_j'}{\partial x}
\end{aligned} \tag{8}
$$

将式(7)和(8)代入式(2)则有:

$$
\begin{aligned}
&\varepsilon\left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x}\right)\frac{\partial\varphi_j'}{\partial y} + \varepsilon\left(f_0^2 v_j' + \beta_0 y\frac{\partial\varphi_j'}{\partial x} - f_0\frac{\partial\varphi_j'}{\partial x}\right) \\
&+ \varepsilon^2\left(u_j'\frac{\partial}{\partial x} + v_j'\frac{\partial}{\partial y}\right)\frac{\partial\varphi_j'}{\partial y} = 0 \\
&\varepsilon\left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x}\right)\frac{\partial\varphi_j'}{\partial x} + \varepsilon\left(f_0^2 u_j' + f_0\frac{\partial\varphi_j'}{\partial y} - \beta_0\frac{\partial\varphi_j'}{\partial y}\right) \\
&+ \left(f_0^2\bar{u}_j - \beta_0 y\frac{\partial\bar{\varphi}_j}{\partial y} + f_0\frac{\partial\varphi_j'}{\partial y}\right) + \varepsilon^2\left(u_j'\frac{\partial}{\partial x} + v_j'\frac{\partial}{\partial y}\right)\frac{\partial\varphi_j'}{\partial x} = 0 \\
&\varepsilon\left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x}\right)\varphi_j' + \varepsilon^2\left(u_j'\frac{\partial}{\partial x} + v_j'\frac{\partial}{\partial y}\right)\varphi_j' + \varepsilon v_j'\frac{\partial\bar{\varphi}_j}{\partial y} \\
&+ \varepsilon C_0^2\left(\frac{\partial u_j'}{\partial x} + \frac{\partial v_j'}{\partial y}\right) = 0
\end{aligned} \tag{9}
$$

式中 $j = 1,2, C_0^2 = g^* H_0, H_0$ 为静态流体深度。

引入 Garden Morikawa 变换[5],即令

$$
\xi = \varepsilon^{1/2}(x - Ct), \tau = \varepsilon^{3/2}t, y = y \tag{10}
$$

其中 C 为待定的扰动速度,为实数。对u_j',v_j' 和 φ_j' 采用小参数展开为:

$$
\begin{aligned}
&u_j' = u_j^{(0)} + \varepsilon u_j^{(1)} + \varepsilon^2 u_j^{(2)} + \cdots \\
&v_j' = \varepsilon^{1/2} v_j^{(0)} + \varepsilon^{3/2} v_j^{(1)} + \varepsilon^{5/2} v_j^{(2)} + \cdots \\
&\varphi_j' = \varphi_j^{(0)} + \varepsilon\varphi_j^{(1)} + \varepsilon^2\varphi_j^{(2)} + \cdots
\end{aligned} \tag{11}
$$

将式(10)和(11)代入式(9)就可得到零级近似方程组

$$
f_0^2\bar{u}_j - \beta_0 y\frac{\partial\bar{\varphi}_j}{\partial y} + f_0\frac{\partial\bar{\varphi}_j}{\partial y} = 0 \qquad (j = 1,2) \tag{12}
$$

一级近似方程组为:

$$(\bar{u}_j - C)\frac{\partial^2 \varphi_j^{(0)}}{\partial\xi 2y} + f_0^2 v_j^{(0)} + \beta_0 y\frac{\partial \varphi_j^{(0)}}{\partial\xi} - f_0\frac{\partial \varphi_j^{(0)}}{\partial\xi} = 0$$

$$f_0^2 u_j^{(0)} + f_0\frac{\partial \varphi_j^{(0)}}{\partial y} - \beta_0 y\frac{\partial \varphi_j^{(0)}}{\partial y} = 0 \tag{13}$$

$$(\bar{u}_j - C)\frac{\partial \varphi_j^{(0)}}{\partial\xi} + v_j^{(0)}\frac{\partial\bar{\varphi}_j}{\partial y} + C_0^2\left(\frac{\partial u_j^{(0)}}{\partial\xi} + \frac{\partial v_j^{(0)}}{\partial y}\right) = 0$$

式中 $j=1,2$

二级近似方程组为：

$$\frac{\partial}{\partial\tau}\left(\frac{\partial \varphi_j^{(0)}}{\partial y}\right) + (\bar{u}_j - C)\frac{\partial^2 \varphi_j^{(1)}}{\partial\xi\partial y} + f_0^2 v_j^{(1)} + \beta_0 y\frac{\partial \varphi_j^{(1)}}{\partial\xi}$$

$$- f_0\frac{\partial \varphi_j^{(1)}}{\partial\xi} + \left(u_j^{(0)}\frac{\partial}{\partial\xi} + v_j^{(0)}\frac{\partial}{\partial y}\right)\frac{\partial \varphi_j^{(0)}}{\partial y} = 0$$

$$(\bar{u}_j - C)\frac{\partial^2 \varphi_j^{(0)}}{\partial\xi^2} + f_0^2 u_j^{(1)} + f_0\frac{\partial \varphi_j^{(1)}}{\partial y} - \beta_0 y\frac{\partial \varphi_j^{(1)}}{\partial y} = 0 \tag{14}$$

$$\frac{\partial \varphi_j^{(0)}}{\partial\tau} + (\bar{u}_j - C)\frac{\partial \varphi_j^{(1)}}{\partial\xi} + \left(u_j^{(0)}\frac{\partial \varphi_j^{(0)}}{\partial\xi} + v_j^{(0)}\frac{\partial \varphi_j^{(0)}}{\partial y}\right)$$

$$+ v_j^{(1)}\frac{\partial\bar{\varphi}_j}{\partial y} + C_0^2\left(\frac{\partial u_j^{(1)}}{\partial\xi} + \frac{\partial v_j^{(1)}}{\partial y}\right) = 0$$

式中 $j=1,2$。

利用式(7)则非线性的动力学边界条件为：

当 $y=\eta(x,t)$ 时，

$$\left(\frac{\partial}{\partial t} + \bar{u}_j\frac{\partial}{\partial x} + \varepsilon u_j'\frac{\partial}{\partial x}\right)(\varphi_1' - \varphi_2') + v_j'\frac{\partial(\bar{\varphi}_1 - \bar{\varphi}_2)}{\partial y} = 0 \tag{15}$$

式中 $j=1,2$。利用式(10)和(11)，由式(15)可得到非线性动力学边界条件的各级近似方程，我们只需讨论零级近似的动力学边界条件：

当 $y=\eta(\xi,\tau)$ 时，

$$(\bar{u}_j - C)\frac{\partial(\varphi_1^{(0)} - \varphi_2^{(0)})}{\partial\xi} + v_j^{(0)}\frac{\partial(\bar{\varphi}_1 - \bar{\varphi}_2)}{\partial y} = 0 \tag{16}$$

4 零级、一级近似方程解

由零级近似方程式(12)有

$$\frac{\partial\bar{\varphi}_j}{\partial y} = \frac{f_0^2\bar{u}_j}{\beta_0 y - f_0} \tag{17}$$

由一级近似方程式(13)有

$$(\bar{u}_j - C)\frac{\partial^3 \varphi_j^{(0)}}{\partial\xi\partial y^2} + \frac{\lambda_0^2\bar{u}_j(\bar{u}_j - C)}{\beta_0 y - f_0}\frac{\partial^2 \varphi_j^{(0)}}{\partial\xi\partial y} + (\beta_0 + C\lambda_0^2)\frac{\partial \varphi_j^{(0)}}{\partial\xi} = 0 \tag{18}$$

式中 $j=1,2$。$\lambda_0^2 = f_0^2/C_0^2$。

将式(18)对 ξ 积分，取积分常数为零，则有

$$\frac{\partial^2 \varphi_j^{(0)}}{\partial y^2} + \frac{\lambda_0^2\bar{u}_j}{\beta_0 y - f_0}\frac{\partial \varphi_j^{(0)}}{\partial y} + \frac{\beta_0 + C\lambda_0^2}{\bar{u}_j - C}\varphi_j^{(0)} = 0 \tag{19}$$

若取

$$\varphi_j^{(0)} = G_j(y)A_j(\xi,\tau) \tag{20}$$

则由式(20)代入式(19)有

$$\frac{\partial^2 G_j}{\partial y^2} + \frac{\lambda_0^2\bar{u}_j}{\beta_0 y - f_0}\frac{\partial^2 G_j}{\partial y} + \frac{(\beta_0 + C\lambda_0^2)}{\bar{u}_j - C}G_j = 0 \tag{21}$$

由自然边界条件式(4)有

$$G_j(y)|_{y\to-\infty} = 0 \qquad G_j(y)|_{y\to+\infty} = 0 \tag{22}$$

因 $\beta_0 y \ll f_0$,则有 $\beta_0 y - f_0 \approx - f_0$,求解式(21)得到:

$$\begin{aligned} G_1 &= k_1 e^{\lambda_1 y} \qquad (-\infty < y \leqslant \eta(\xi,\tau)) \\ G_2 &= k_2 e^{-\lambda_2 y} \qquad (\eta(\xi,\tau) \leqslant y < +\infty) \end{aligned} \tag{23}$$

式中 k_1,k_2 为积分常数。

$$\lambda_1 = \frac{\lambda_0^2\bar{u}_1}{2f_0} + \frac{1}{2}\sqrt{\frac{\lambda_0^4\bar{u}_1^2}{f_0^2} - \frac{4(\beta_0 + C\lambda_0^2)}{\bar{u}_1 - C}}$$

$$\lambda_2 = \frac{\lambda_0^2\bar{u}_2}{2f_0} + \frac{1}{2}\sqrt{\frac{\lambda_0^4\bar{u}_2^2}{f_0^2} - \frac{4(\beta_0 + C\lambda_0^2)}{\bar{u}_2 - C}}$$

将式(23)代入式(20)则有

$$\begin{aligned} \varphi_1^{(0)} &= k_1 e^{\lambda_1 y}A_1(\xi,\tau) \\ \varphi_2^{(0)} &= k_2 e^{-\lambda_2 y}A_2(\xi,\tau) \end{aligned} \tag{24}$$

5　二级近似与 KDV 方程

在切变线 $y = \eta(\xi,\tau)$,利用式(12),动力学边界条件式(16)可以写为

$$(\bar{u}_j - C)\frac{\partial(\varphi_1^{(0)} - \varphi_2^{(0)})}{\partial\xi} + v_j^{(0)}\frac{f_0^2(\bar{u}_1 - \bar{u}_2)}{\beta_0 y - f_0} = 0 \tag{25}$$

令式中 $j = 1$ 和 $j = 2$,则由式(25)有

$$v_1^{(0)} = \frac{\beta_0 y - f_0}{f_0^2(\bar{u}_1 - \bar{u}_2)}(\bar{u}_1 - C)\frac{\partial(\varphi_1^{(0)} - \varphi_2^{(0)})}{\partial\xi} \tag{26}$$

$$v_2^{(0)} = -\frac{\beta_0 y - f_0}{f_0^2(\bar{u}_1 - \bar{u}_2)}(\bar{u}_1 - C)\frac{\partial(\varphi_1^{(0)} - \varphi_2^{(0)})}{\partial\xi} \tag{27}$$

由一级近似方程式(13)中的第一式,令 $j = 2$ 有:

$$v_2^{(0)} = -\frac{\bar{u}_2 - C}{f_0^2}\frac{\partial^2\varphi_2^{(0)}}{\partial\xi\partial y} - \frac{\beta_0 y - f_0}{f_0^2}\frac{\partial\varphi_2^{(0)}}{\partial\xi} \tag{28}$$

因 $\beta_0 y \ll f_0$,取 $\beta_0 y - f_0 \approx - f_0$,并利用式(24),则由式(27)和(28)有

$$\frac{\partial A_2}{\partial\xi} = \frac{k_1 f_0(\bar{u}_2 - C)e^{(\lambda_1+\lambda_2)y}}{k_2(\bar{u}_2 - C)(\bar{u}_1 - \bar{u}_2)\lambda_2 + k_2 f_0(\bar{u}_1 - C)}\frac{\partial A_1}{\partial\xi} \tag{29}$$

利用式(29),将式(24)代入式(26)式则有

$$\begin{aligned} v_1^{(0)} &= \frac{[f_0 + (\bar{u}_2 - C)\lambda_2](\bar{u}_1 - C)k_1 e^{\lambda_1 y}}{f_0[f_0(\bar{u}_1 - C) + (\bar{u}_1 - \bar{u}_2)(\bar{u}_2 - C)\lambda_2]}\frac{\partial A_1}{\partial\xi} \\ \frac{\partial v_1^{(0)}}{\partial y} &= \frac{\lambda_1[f_0 + (\bar{u}_2 - C)\lambda_2](\bar{u}_1 - C)k_1 e^{\lambda_1 y}}{f_0[f_0(\bar{u}_1 - C) + (\bar{u}_1 - \bar{u}_2)(\bar{u}_2 - C)\lambda_2]}\frac{\partial A_1}{\partial\xi} \end{aligned} \tag{30}$$

在以下方程的讨论中均取 $j=1$。

由一级近似方程组式(13)中的第二式并利用式(24)取 $\beta_0 y - f_0 \approx - f_0$，则有

$$
\begin{aligned}
u_1^{(0)} &= -\frac{\lambda_1}{f_0} k_1 e^{\lambda_1 y} A_1 \\
\frac{\partial u_1^{(0)}}{\partial y} &= -\frac{\lambda_1^2}{f_0} k_1 e^{\lambda_1 y} A_1
\end{aligned} \tag{31}
$$

由式(30)和(31)可知

$$
\begin{aligned}
&\left(\frac{\partial u_1^{(0)}}{\partial y}\lambda_1 \frac{\partial A_1}{\partial \xi} + \frac{\partial v_1^{(0)}}{\partial y}\lambda_1^2 A_1\right) k_1 e^{\lambda_1 y} - \frac{\beta_0 + \bar{u}_1 \lambda_0^2}{\bar{u}_1 - C}\left(u_1^{(0)} \frac{\partial A_1}{\partial \xi} + v_1^{(0)} \lambda_1 A_1\right) k_1 e^{\lambda_1 y} \\
&= \frac{(\bar{u}_2 - C)^2 \lambda^2}{f_0(\bar{u}_1 - C) + (\bar{u}_1 - \bar{u}_2)(\bar{u}_2 - C)\lambda_2} \cdot \frac{\lambda_1}{f_0} \cdot \left(\lambda_1^2 - \frac{\beta_0 + \bar{u}_1 \lambda_0^2}{\bar{u}_1 - C}\right) \\
&\cdot (k_1 e^{\lambda_1 y})^2 A_1 \frac{\partial A_1}{\partial \xi}
\end{aligned} \tag{32}
$$

由二级近似方程组式(14)有

$$
\begin{aligned}
&(\bar{u}_j - C) \frac{\partial}{\partial \xi}\left[\frac{\partial^2 \varphi_j^{(1)}}{\partial y^2} + \frac{\lambda_0^2 \bar{u}_j}{\beta_0 y - f_0} \frac{\partial \varphi_j^{(1)}}{\partial y} + \frac{\beta_0 + C\lambda_0^2}{\bar{u}_j - C} \varphi_j^{(1)}\right] \\
&+ (\bar{u}_j - C) \frac{\partial^3 \varphi_j^{(0)}}{\partial \xi^3} - \frac{\beta_0 + \bar{u}_j \lambda_0^2}{\bar{u}_j - C} \frac{\partial \varphi_j^{(0)}}{\partial \tau} + \left(\frac{\partial u_j^{(0)}}{\partial y} \frac{\partial}{\partial \xi}\right. \\
&\left. + \frac{\partial v_j^{(0)}}{\partial y} \frac{\partial}{\partial y}\right) \frac{\partial \varphi_j^{(0)}}{\partial y} - \frac{\beta_0 + \bar{u}_j \lambda_0^2}{\bar{u}_j - C}\left(u_j^{(0)} \frac{\partial}{\partial \xi} + v_j^{(0)} \frac{\partial}{\partial y}\right) \varphi_j^{(0)} = 0
\end{aligned} \tag{33}
$$

式中 $j=1,2$。

对式(33)取 $j=1$，并将式(24)代入，利用关系式(32)则有：

$$
\begin{aligned}
&(\bar{u}_1 - C) \frac{\partial}{\partial \xi}\left[\frac{\partial^2 \varphi_1^{(1)}}{\partial y^2} + \frac{\lambda_0^2 \bar{u}_1}{\beta_0 y - f_0} \frac{\partial \varphi_1^{(1)}}{\partial y} + \frac{\beta_0 + C\lambda_0^2}{\bar{u}_1 - C} \varphi_1^{(1)}\right] \\
&- \frac{\beta_0 + \bar{u}_1 \lambda_0^2}{\bar{u}_1 - C} k_1 e^{\lambda_1 y} \frac{\partial A_1}{\partial \tau} + (\bar{u}_1 - C) k_1 e^{\lambda_1 y} \frac{\partial^3 A_1}{\partial \xi^3} \\
&+ \frac{(\bar{u}_2 - C)^2 \lambda_2}{f_0(\bar{u}_1 - C) + (\bar{u}_1 - \bar{u}_2)(\bar{u}_2 - C)\lambda_2} \cdot \frac{\lambda_1}{f_0} \cdot \left(\lambda_1^2 - \frac{\beta_0 + \bar{u}_1 \lambda_0^2}{\bar{u}_1 - C}\right) \\
&\cdot (k_1 e^{\lambda_1 y})^2 \cdot A_1 \frac{\partial A_1}{\partial \xi} = 0
\end{aligned} \tag{34}
$$

将式(34)乘以 G_1 并对 y 积分，从 $-\infty$ 积分到切变线 $y=\eta(\xi,\tau)$，并利用边界条件有

$$
\begin{aligned}
&\frac{\partial}{\partial \xi}\int_{-\infty}^{\eta(\xi,\tau)} G_1\left[\frac{\partial^2 \varphi_1^{(1)}}{\partial y^2} + \frac{\lambda_0^2 \bar{u}_1}{\beta_0 y - f_0} \frac{\partial \varphi_1^{(1)}}{\partial y} + \frac{\beta_0 + C\lambda_0^2}{\bar{u}_1 - C} \varphi_1^{(1)}\right] \mathrm{d}y \\
&= \frac{\partial}{\partial \xi}\int_{-\infty}^{\eta(\xi,\tau)} \frac{\partial}{\partial y}\left(G_1 \frac{\partial \varphi_1^{(1)}}{\partial y} - \varphi_1^{(1)} \frac{\partial G_1}{\partial y}\right) \mathrm{d}y \\
&+ \frac{\partial}{\partial \xi}\int_{-\infty}^{\eta(\xi,\tau)} \varphi_1^{(1)}\left[\frac{\partial^2 G_1}{\partial y^2} + \frac{\lambda_0^2 \bar{u}_1}{\beta_0 y - f_0} \frac{\partial G_1}{\partial y} + \frac{\beta_0 + C\lambda_0^2}{\bar{u} - C} G_1\right] \mathrm{d}y \\
&+ \frac{\lambda_0^2 \bar{u}_1}{\beta_0 y - f_0} \frac{\partial}{\partial \xi}\int_{-\infty}^{\eta(\xi,\tau)} \frac{\partial}{\partial y}\left(G_1 \frac{\partial \varphi_1^{(1)}}{\partial y} - \varphi_1^{(1)} \frac{\partial G_1}{\partial y}\right) \mathrm{d}y = 0
\end{aligned} \tag{35}
$$

$$\int_{-\infty}^{\eta(\xi,\tau)}(e^{\lambda_1 y})^2\mathrm{d}y=\frac{1}{2\lambda_1}(e^{\lambda_1\eta 2}) \tag{36}$$

$$\int_{-\infty}^{\eta(\xi,\tau)}(e^{\lambda_1 y})^3\mathrm{d}y=\frac{1}{3\lambda_1}(e^{\lambda_1\eta 3}) \tag{37}$$

最后求得

$$\frac{\partial A}{\partial\tau}+RA\frac{\partial A}{\partial\xi}+S\frac{\partial^3 A}{\partial\xi^3}=0 \tag{38}$$

式中

$$\begin{aligned}R&=-\frac{2}{3}k_1\cdot\frac{\lambda_1}{f_0}\cdot\frac{1}{\beta_0+\bar{u}_1\lambda_0^2}\cdot\frac{(\bar{u}_1-C)(\bar{u}_2-C)}{f_0(\bar{u}_1-C)+(\bar{u}_1-\bar{u}_2)(\bar{u}_2-C)\lambda_2}\\&\quad\cdot(\lambda_1^2-\frac{\beta_0+\bar{u}_1\lambda_0^2}{\bar{u}_1-C})\\S&=\frac{(\bar{u}_1-C)^2}{\beta_0+\bar{u}_1\lambda_0^2}\\A&=A_1e^{\lambda_1\eta}\end{aligned} \tag{39}$$

由式(38)可以看出扰动的振幅 A 满足 KDV 方程。

6　KDV 方程与孤立波

我们不难求出 KDV 方程式(38)的椭圆余弦波解[6]为

$$A(\xi,\tau)=\alpha Cn^2\sqrt{\frac{R}{12S}}\beta(\xi-C\tau) \tag{40}$$

式中

$$\alpha=\frac{3C}{2R}\pm\frac{1}{2}\sqrt{\frac{9C^2}{R^2}+\frac{24S}{R}k_3}$$

$$\beta=\pm\sqrt{\frac{9C^2}{R^2}+\frac{24S}{R}k_3}$$

$$\begin{aligned}\frac{R}{S}&=\frac{2}{3}k_1\cdot\frac{\lambda_1}{f_0(\bar{u}_1-C)}\cdot\frac{u_2-C}{f_0(\bar{u}_1-C)+(\bar{u}_1-\bar{u}_2)(\bar{u}_2-C)\lambda_2}\\&\quad\cdot(\lambda_1^2-\frac{\beta_0+\bar{u}_1\lambda_0^2}{\bar{u}_1-C})\end{aligned}$$

其中 k_3 为积分常数。

当 $\alpha=\beta$ 时,即积分常数 $k_3=0$ 时则有

$$\alpha=\beta=\frac{3C}{R} \tag{41}$$

则椭圆余弦波解退化为孤立波解:

$$A(\xi,\tau)=\alpha\mathrm{sech}^2\sqrt{\frac{C}{4S}}(\xi-C\tau) \tag{42}$$

其中的波数 C 由下式确定

$$C=\frac{R\alpha}{3} \tag{43}$$

由尺度分析可知

$$\frac{\lambda_0^2\bar{u}_1^2}{f_0^2} \gg \frac{4\beta_0 + C\lambda_0^2}{\bar{u}_1 - C}$$

由前面 λ_1 和 λ_2 的表达式可以认为 λ_1 和 λ_2 是与 C 无关的常数。

将 R 的表达式代入式(43) 则有关于 C 的一元二次代数方程

$$aC^2 + bC + d = 0 \tag{44}$$

式中

$$a = \frac{2\alpha k_1\lambda_1^3}{9f_0(\beta_0 + \bar{u}_1\lambda_0^2)} - f_0 - \lambda_2(\bar{u}_1 - \bar{u}_2)$$

$$b = \frac{2\alpha k_1\lambda_1[(\bar{u}_1 + \bar{u}_2)\lambda_1^2 - (\beta_0 + \bar{u}_1\lambda_0^2)]}{9f_0(\beta_0 + \bar{u}_1\lambda_0^2)} + f_0\bar{u}_1 + (\bar{u}_1 - \bar{u}_2)\lambda_2\bar{u}_2$$

$$d = \frac{2\alpha k_1\lambda_1[\lambda_1^2\bar{u}_1\bar{u}_2 - (\beta_0 + \bar{u}_1\lambda_0^2)\bar{u}_2]}{9f_0(\beta_0 + \bar{u}_1\lambda_0^2)}$$

由式(44)可以求出波数 C 的实数解。由一元二次代数方程的理论可以求出出现孤立波的水平风速切变条件为:

$$(\bar{u}_1 - \bar{u}_2)^2 + \frac{4(d - \bar{u}_2B_2)}{\bar{u}_2^2\lambda_2}(\bar{u}_1 - \bar{u}_2) + \frac{\beta_2^2 + 4B_1d}{\lambda_2^2\bar{u}_2^2} > 0 \tag{45}$$

式中

$$B_1 = \frac{2\alpha k_1\lambda_1^3}{9f_0(\beta_0 + \bar{u}_1\lambda_0^2)} - f_0$$

$$B_2 = \frac{2\alpha k_1\lambda_1[(\bar{u}_1 + \bar{u}_2)\lambda_1^2 - (\beta_0 + \bar{u}_1\lambda_0^2)]}{9f_0(\beta_0 + \bar{u}_1\lambda_0^2)} + f_0\bar{u}_1$$

由式(45)有

$$\bar{u}_1 - \bar{u}_2 > -\frac{2(d - B_2\bar{u}_2)}{\lambda_2\bar{u}_2} + \sqrt{\frac{(d - B_2\bar{u}_2)^2}{\lambda_2^2\bar{u}_2^4} - \frac{B_2^2 + 4B_1d}{\lambda_2^2\bar{u}_2^2}}$$

或且 (46)

$$\bar{u}_1 - \bar{u}_2 < -\frac{2(d - B_2\bar{u}_2)}{\lambda_2\bar{u}_2} + \sqrt{\frac{(d - B_2\bar{u}_2)^2}{\lambda_2^2\bar{u}_2^4} - \frac{B_2^2 + 4B_1d}{\lambda_2^2\bar{u}_2^2}}$$

当 $\alpha \to 0$ 时,KDV 方程被线性化为:

$$\frac{\partial A}{\partial \tau} = \frac{\partial^3 A}{\partial \xi^3} \tag{47}$$

若令其解为 $A' = \overline{A}e^{k(\xi - C\tau)}$ 有

$$\begin{aligned} &\text{相速度} \qquad C = -k^2 \\ &\text{群速度} \qquad C_g = -3k^2 \end{aligned} \tag{48}$$

这是向西传播的左行波。

由此可知,孤立波仅是椭圆余弦波的一个特例。对非线性项的研究表明,非线性的作用使方程的解出现不连续(即出现弱解),使波变陡,从而导致激波的出现,而色散性的作用则使要突陡的波的各个部分以不同的速度被色散而传播,消弱非线性引起的波的突陡,当非线性作用和色散作用达到平衡时,即形成孤立波。由式

$$\frac{R}{S}=\frac{2k_1}{3}\cdot\frac{\lambda_1}{f_0(\bar{u}_1-C)}\cdot\frac{(\bar{u}_2-C)}{f_0(\bar{u}_1-C)+(\bar{u}_1-\bar{u}_2)(\bar{u}_2-C)\lambda_2}$$
$$\cdot\left(\lambda_1^2-\frac{\beta_0+\bar{u}_1\lambda_0^2}{\bar{u}_1-C}\right)$$

也可看出，β效应和水平辐合辐散对非线性项的存在起着突出的作用。因此，水平风速切变对孤立波的出现具有重要意义。也正因为非线性项的作用，使切变线上出现了椭圆余弦波或孤立波。

我们知道，在等压面上的低涡表现为气压要素的等值线的中心密集区，外围稀散区。沿 y 方向取一剖线并以该剖线为横坐标，沿剖线的要素梯度为纵坐标则可绘出要素梯度在 y 方向的分布图，在 β 平面近似下，设低涡外围区气压梯度值为常数，理想状态下的分布为一典型的孤立波（图略），同样，沿 x 方向和垂直方向可以看到类似的孤立波，因而低涡在空间上可以看成是三维孤立波。

我们是从切变线上描述涡旋运动的方程出发的，而低涡又可看成孤立波，因而水平风速切变线上的孤立波可以形成低涡。由此可见，水平风速切变线上低涡的形成是由于非线性作用的结果，即非线性过程和色散过程共同作用并达到平衡而形成的。这不仅在理论上较好地解释了为什么切变线是一个多波动且易生成低涡的天气系统。预报实践告诉我们，过去那种一贯只注意从上游移来的低值系统沿切变线移动，而不太着重切变线本身对低涡的生成作用的预报观点是不全面的。

参考文献

[1] 杨大升等．动力气象学．北京：气象出版社．1981．272—276．

[2] 刘式适，刘式达．半地转近似下的非线性波动．气象学报，1987，45：257—260．

[3] Keyser D and Pecnik M J. The effect of along-front temperature variation in a two-dimensional primitive equation model of surface frontogenesis . J Atmos Sci, 1987, 44: 577－604.

[4] 李麦村．大气中飑线形成的非线性过程与 KDV 方程．中国科学，1981，3：341—346．

[5] 谷内俊弥．非线性波动，徐福元等译．北京：原子能出版社，1981．

[6] 刘式适，刘式达．大气动力学．1991，北京大学出版社，330—338．

STABILITY OF THE NONLINEAR WAVES ON THE HORIZONTAL SHEAR-LINE OF WIND WITH THE GEOSTROPHIC MOMENTUM APPROXIMATION

Li Ziliang Wan Jun

(CAAC Flying College, Guonghan, 618307) (Chengdu Institute of Meteorology, Chengdou)

Abstract

In this paper, stability of the nonlinear waves that are produced by horizontal shearing of wind in the geostrophic momentum approximation two-level atmesphric model are discussed. The solution of solitary is obtained. The analytical solution of the nonlinear equation and nonlinear boundary conditions explains well the weather phenomena that vortex is easy to produce on the horizontal shear-line of wind.

Key words: Geostrophic momentum approximation, WKB method, KDV equation, Solitary wave.

中国气象局气科院情报所关于“查新咨询”工作的启事

中国气象局气科院情报所 1993 年被中国气象局批准为气象部门查新咨询单位.

近年来, 中国气象局气科院情报所承接了相当数量的查新课题, 进行了多项科研立题、科技成果鉴定、科技奖、专利申请、产品开发及技术咨询等的查新工作.

情报所图书馆有极其丰富的馆藏: 期刊近 3 万册, 其中外文期刊 2 万多册; 图书近 10 万册, 其中外文图书近 7 万册, 而且大多为原版期刊和图书. 通过多年实践, 积累了一定的经验, 培养并锻炼了专门从事科技查新咨询的人员, 制定及完善了查新咨询工作流程, 并严格了科技咨询的管理办法.

最近, 情报所正在建立机检系统, 引进了美国的 GMA (气象和天体物理文摘) 光盘, 可以检索从 1974 年起至今的国外气象刊物报道的有关文献, 预计 1995 年 9 月份起可正式开展服务. 欢迎气象部门广大科技工作者前来查新咨询.

业务联系电话: 8321133—2204

联系人: 周景林